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Averaging Method Analysis of Synchronization Characteristics of a Large Number of Nonlinearly Coupled van der Pol Oscillators

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SUMMARY The averaged equation for an arbitrary number of oscillators coupled by nonlinear coupling scheme invented by S. Nagano, is derived. This system is invented as a model of uni-cellular slime amoeba. By using the averaged equation, we investigate the synchronization characteristics of five coupled oscillators and a large number of coupled oscillators. In particular, we present the statistical property of coupled oscillators in terms of coupling factor γ. We also investigate the effect of linear and nonlinear coupling terms for achieving synchronization, and confirm that the nonlinear coupling term plays an important role for strong synchronization than linear coupling term does.

key words: *synchronization, coupled oscillator system, averaging method, nonlinear coupling, slime amoeba*

1. Introduction

The study of systems of coupled oscillators has attracted constant interest in various areas of engineering, physics, and mathematics [1]–[4]. In particular, mutual synchronization of rhythms is extremely significant due to its practical needs [5], [6]. Coupled oscillator system are often used in modeling biological systems [7]. Recently, S. Nagano invented a new nonlinear coupling scheme between limit cycle oscillators based on a model of an aggregated uni-celluar slime amoeba called "Dictyostelium discoideum" [8], [9]. He employed a van der Pol oscillator as a typical model of the limit cycle oscillator. One of the characteristics of this nonlinear coupling scheme is its strong same-phase synchronization ability. Namely, different kind of oscillators with different natural frequencies can be synchronized easily. Although these characteristics have been confirmed by direct computer simulation, there is no theoretical approach to this system so far.

In our previous paper, we analyzed two coupled van der Pol oscillators with this scheme via averaging method, and clarified various interesting characteristics peculiar to nonlinear coupling [10]. In this paper, we are succeeded in analyzing an "arbitrary" number of van der Pol oscillators coupled by this scheme via averaging method. Namely, the explicit form of averaged equation for an "arbitrary" number of coupled van der Pol oscillators can be derived, and by using this equation, the steady-state characteristics of the same-phase synchronized solution (mode) can be obtained. The reason why we concentrate on the same-phase mode is because it is practically important and indeed, the uni-celluar slime amoeba shows the same-phase synchronization to communicate with each other during aggregation [11]*. Although the averaging method can be applicable to "weak" nonlinear case theoretically, it is confirmed that in our case the results of the averaging method (synchronization probability versus the coupling factor) are applicable for considerably strong nonlinear case.

2. Derivation of the Averaged Equation

A van der Pol oscillator adopted in this system is written as follows (Appendix A).

$$\frac{d^2x}{dt^2} - \varepsilon\omega(1-x^2)\frac{dx}{dt} + \omega^2 x = 0 \tag{1}$$

where ε denotes a (small positive) parameter showing the degree of nonlinearity and where ω denotes the natural angular frequency**.

Coupling the N-van der Pol oscillators via Nagano's method gives the following equation [9]:

$$\ddot{x}_k + \omega_k^2\left(x_k + \gamma_k\sum_{l=1}^{N} x_l\right) = \varepsilon\omega_k\left\{1-\left(x_k + \gamma'_k\sum_{l=1}^{N} x_l\right)^2\right\}\dot{x}_k, k = 1, 2, \ldots, N \tag{2}$$

where $\gamma_k > 0$ denotes a linear coupling factor, and where $\gamma'_k > 0$ denotes a nonlinear coupling factor. Equation (2) can be rewritten in the following vector form:

$$\ddot{\mathbf{x}} + \mathbf{B}\mathbf{x} = \varepsilon\mathbf{C}\dot{\mathbf{x}} - \varepsilon G(\mathbf{x}, \dot{\mathbf{x}}) \tag{3a}$$

where

$$\mathbf{x} \equiv [x_1, x_2, \ldots, x_N]^T \tag{3b}$$

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*More accurately, each time waveform of the intercellular cAMP (= cyclic adenosine 3′, 5′-monophosphate) density presents the same-phase synchronization during aggregation.

**In Nagano's model, x associates with the intercellular cAMP product, and $\dot{x}$ with the cAMP receptor.

$$\mathbf{B} \equiv \begin{bmatrix} (1+\gamma_1)\omega_1^2 & \gamma_1\omega_1^2 & \cdots\cdots\cdots & \gamma_1\omega_1^2 \\ \gamma_2\omega_2^2 & (1+\gamma_2)\omega_2^2 & \gamma_2\omega_2^2 & \cdots\cdots & \gamma_2\omega_2^2 \\ \vdots & & \ddots & & \vdots \\ \gamma_N\omega_N^2 & \cdots\cdots\cdots\cdots & \gamma_N\omega_N^2 & (1+\gamma_N)\omega_N^2 \end{bmatrix} \tag{3c}$$

$$\mathbf{C} \equiv \begin{bmatrix} \omega_1 & & 0 \\ & \ddots & \\ 0 & & \omega_N \end{bmatrix} \tag{3d}$$

$$G(\mathbf{x}, \dot{\mathbf{x}}) \equiv [g_1(\mathbf{x}, \dot{\mathbf{x}}), g_2(\mathbf{x}, \dot{\mathbf{x}}), \ldots, g_N(\mathbf{x}, \dot{\mathbf{x}})]^T . \tag{3e}$$

Each component of $G(\mathbf{x}, \dot{\mathbf{x}})$ can be written as follows.

$$g_k(\mathbf{x}, \dot{\mathbf{x}}) = \left(x_k + \gamma_k' \sum_{l=1}^{N} x_l \right)^2 \omega_k \dot{x}_k \tag{4}$$

We define the eigenvalues of $\mathbf{B}$ as $\Omega_1^2 < \Omega_2^2 < \cdots < \Omega_N^2$ and the associated eigenvectors as $\mathbf{p_j} = [p_{1j}, p_{2j}, \cdots, p_{Nj}]^T$ for $j = 1, 2, \ldots, N$, respectively†. Applying a non-singular linear transformation $\mathbf{x} = \mathbf{Py}$ to Eq. (3a) and multiplying $\mathbf{P^{-1}}$ from the left-hand side give the following transformed equation.

$$\ddot{\mathbf{y}} + \tilde{\mathbf{B}}\mathbf{y} = \varepsilon\tilde{\mathbf{C}}\dot{\mathbf{y}} - \varepsilon\mathbf{Q}G(\mathbf{y}, \dot{\mathbf{y}}) \tag{5a}$$

where

$$\tilde{\mathbf{B}} \equiv \mathbf{P^{-1}BP} = \begin{bmatrix} \Omega_1^2 & & 0 \\ & \ddots & \\ 0 & & \Omega_N^2 \end{bmatrix}$$

$$\tilde{\mathbf{C}} \equiv \mathbf{P^{-1}CP} = [\tilde{c}_{ij}]$$

$$\mathbf{P} = [p_{ij}]$$

$$\mathbf{Q} \equiv \mathbf{P^{-1}} = [q_{ij}],\ i, j = 1, 2, \ldots, N \tag{5b}$$

Equation (5a) can be written in the following scalar form:

$$\begin{aligned} \ddot{y}_k + \Omega_k^2 y_k &= \varepsilon \sum_{l=1}^{N} \tilde{c}_{kl}\dot{y}_k - \varepsilon h_k(\mathbf{y}, \dot{\mathbf{y}}) \\ &\equiv \varepsilon f_k(\mathbf{y}, \dot{\mathbf{y}}),\ k = 1, 2, \ldots, N \end{aligned} \tag{6a}$$

where the functions h_k is expressed in terms of $\mathbf{y}, \dot{\mathbf{y}}$ through the transformation $\mathbf{x} = \mathbf{Py}$ as follows (Appendix B) :

$$h_k(\mathbf{y}, \dot{\mathbf{y}}) = \sum_{m_1=1}^{N} \sum_{m_2=1}^{N} \sum_{m_3=1}^{N} b\,(k, m_1, m_2, m_3) y_{m_1} y_{m_2} \dot{y}_{m_3} \tag{6b}$$

and where

$$b\,(k, m_1, m_2, m_3) \equiv \sum_{s=1}^{N} q_{ks}\omega_s a_{sm_1} a_{sm_2} p_{sm_3}$$

$$a_{sm} = p_{sm} + \gamma_s' \sum_{l=1}^{N} p_{lm} \tag{6c}$$

In the averaging method, we first assume $\varepsilon = 0$ in Eq. (6a), and obtain the unperturbed solution. It can be calculated easily as $y_k = \rho_k \sin(\Omega_k t + \theta_k)$, $\dot{y}_k = \rho_k\Omega_k \cos(\Omega_k t + \theta_k)$ for $k = 1, 2, \ldots, N$.

When $\varepsilon \neq 0$, we assume ρ_k and θ_k as functions of time. From the theory of averaging, the dynamics of these functions can be calculated from the following averaged equation for $k = 1, 2, \ldots, N$ [12]:

$$\begin{aligned} \dot{\rho}_k &= \frac{\varepsilon}{\Omega_k} \langle f_k(\mathbf{y}, \dot{\mathbf{y}}) \cos(\Omega_k t + \theta_k) \rangle \\ \dot{\theta}_k &= -\frac{\varepsilon}{\Omega_k \rho_k} \langle f_k(\mathbf{y}, \dot{\mathbf{y}}) \sin(\Omega_k t + \theta_k) \rangle \end{aligned} \tag{7}$$

where $\langle \cdot \rangle$ denotes the time average from zero to infinity. Equation (7) can be calculated by assuming the non-resonant condition for $k = 1, 2, \ldots, N$ as follows (Appendix C):

$$\begin{aligned} \dot{\rho}_k &= \frac{1}{2}\varepsilon\rho_k \left(\tilde{c}_{kk} - \frac{1}{2} \sum_{m_1=1}^{N} b(k, m_1, m_1, k)\rho_{m_1}^2 \right. \\ &\qquad \left. + \frac{1}{4} b(k, k, k, k)\rho_k^2 \right) \\ \dot{\theta}_k &= 0. \end{aligned} \tag{8a}$$

where $b(k, m_1, m_2, m_3)$ is given in Eq. (6c) and $\tilde{c}_{kk}$ is given as

$$\tilde{c}_{kk} = \sum_{l=1}^{N} \omega_l q_{kl} p_{lk} \tag{8b}$$

The non-resonant condition can be written as follows for $m_1, m_2, m_3 = 1, 2, \cdots, N$.

$$\begin{aligned} &(1)\ \Omega_{m_1} \pm \Omega_{m_2} \pm \Omega_{m_3} \pm \Omega_k = 0 \quad \text{for } m_1 \neq m_2 \neq m_3 \neq k \\ &(2)\ 2\Omega_k \pm \Omega_{m_1} \pm \Omega_{m_2} = 0 \quad \text{for } m_1 \neq m_2 \neq k \\ &(3)\ 3\Omega_k - \Omega_{m_3} = 0 \quad \text{for } k \neq m_3 \end{aligned} \tag{9}$$

Since θ_k can be determined as an arbitrary constant, we only investigate the amplitude equation. Therefore, by defining $U_k \equiv \rho_k^2$, Eq.(8a) can be simplified as

$$\begin{aligned} \dot{U}_k &= \varepsilon U_k \left(\tilde{c}_{kk} - \frac{1}{2} \sum_{m_1=1}^{N} b(k, m_1, m_1, k) U_{m_1} \right. \\ &\qquad \left. + \frac{1}{4} b(k, k, k, k) U_k \right) \\ &\equiv \varepsilon F_k(U_1, U_2, \cdots, U_N) \end{aligned} \tag{10}$$

for $k = 1, 2, \ldots, N$. We regard Eq. (10) as the fundamental averaged equation.

†It is not verified so far that the eigenvalues of $\mathbf{B}$ are all positive real numbers. However, as far as our computer calculation shows, they are positive real numbers. It should be clarified in the future that in what case they become positive real numbers.

3. Analysis of Steady-State Solutions via Averaging Method

The steady-state solution corresponds to an equilibrium point in Eq. (10). There are many equilibrium points for a large number of mutually coupled oscillators. The stability of each equilibrium point can be obtained by using the following Jacobian matrix .

$$\mathbf{J} = \begin{bmatrix} \frac{\partial F_1}{\partial U_1} & \cdots & \frac{\partial F_1}{\partial U_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_N}{\partial U_1} & \cdots & \frac{\partial F_N}{\partial U_N} \end{bmatrix} \tag{11}$$

Namely, if all eigenvalues of **J** have negative real parts, the solution is asymptotically stable. If, at least, one of them has positive real part, it is unstable. In this manner, we can judge the stability of each equilibrium point.

By direct computer simulation of Eq. (2), if the nonlinearity is weak (for example, $\varepsilon = 0.1$), we can observe multiple steady state solutions determined from initial conditions. On the other hand, for strong nonlinear case (for example, $\varepsilon = 1.0$), we can observe only the same-phase solution. Besides, each time waveform of the intercellular cAMP density of actual uni-cellulear slime amoeba presents the same-phase synchronization during aggregation [11]. Therefore, we will concentrate on the same-phase solution in this paper.

From Frobenius theory [13], the sign of all components of eigenvector $(p_{kN}, k = 1, 2, \ldots, N)$ associated with the largest eigenvalue (Ω_N^2) of **B** is positive. Therefore, the same-phase solution takes the following form:

$$\begin{aligned} U_N \neq 0, U_k = 0, k = 1, \ldots, N-1 \\ \Rightarrow U_{0N} = \frac{4}{b(N,N,N,N)}, U_{0k}=0 \\ \Rightarrow \rho_{0N} = \sqrt{\frac{4}{b(N,N,N,N)}}, \rho_{0k}=0 \\ \Rightarrow y_N = \rho_{0N}\sin(\Omega_N t + \theta_N), y_k = 0 \\ \Rightarrow x_1 = p_{1N}\rho_{0N}\sin(\Omega_N t + \theta_N) \\ x_2 = p_{2N}\rho_{0N}\sin(\Omega_N t + \theta_N) \\ \vdots \\ x_N = p_{NN}\rho_{0N}\sin(\Omega_N t + \theta_N) \end{aligned} \tag{12}$$

The p_{iN} is positive for $i = 1, 2, \ldots, N$ from Frobenius theory. In Eq. (12), the phase θ_N is an arbitrary constant. Therefore, we take $\dot{\theta}_i = 0$ for all i for simplicity.

4. Synchronization Characteristics of the Same-Phase Solution

In this section, we will show computer calculations of the same-phase synchronization based on the averaged equation. At first, we will investigate the synchronization characteristics of five coupled oscillators. Then, we will investigate a larger number of coupled oscillators. Although we can fix the linear and nonlinear coupling terms γ_k and γ'_k independently, we take these two factors equal (such as $\gamma \equiv \gamma_k = \gamma'_k$ for all k) for most cases in the succeeding sections, unless so identified.

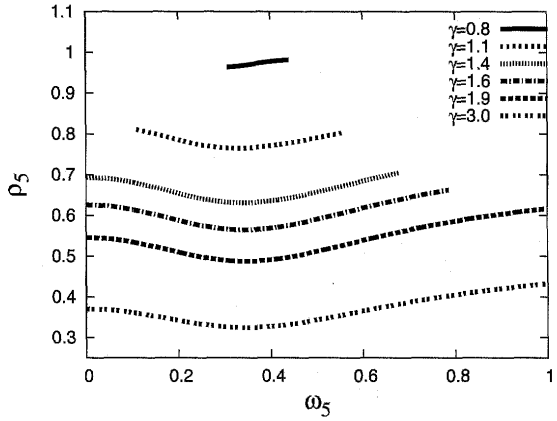


Fig. 1 Steady-state amplitude ρ_5 in terms of ω_5 (varying from 0.001 to 1.0) for various values of coupling factor γ for $\omega_1 = 0.1, \omega_2 = 0.2, \omega_3 = 0.3, \omega_4 = 0.4$.

4.1 Synchronization Characteristics for Five Coupled Oscillators

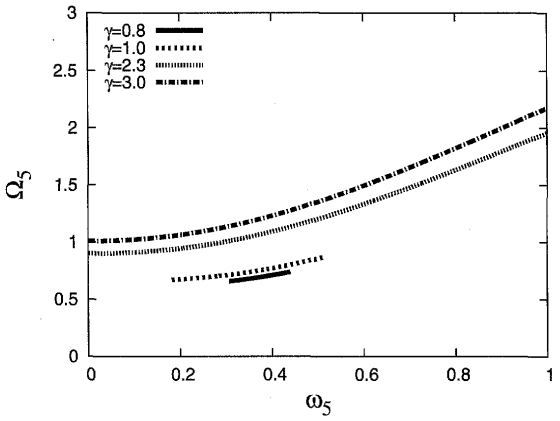
We will investigate the synchronization property of the same-phase solution for five coupled oscillators.

Figure 1 presents the same-phase amplitude ρ_5 (in the y-domain) in terms of $\omega_5 = 0.001$–1.0 for various values of γ, where we fixed $\omega_1 = 0.1, \omega_2 = 0.2, \omega_3 = 0.3$ and $\omega_4 = 0.4$[†]. In this case, the same-phase solution exists for $\gamma \geq 0.8$ only. Both ends of each curve present the synchronization limit. In this system, the synchronization range increases with the increase of coupling factor γ. In particular, for $\gamma \geq \gamma_c \approx 1.9$, synchronization range expands drastically. Namely, synchronization can be achieved for all values of $\omega_5 = 0.001 \sim 1.0$. Such a property could be observed for other cases. For example, when we fix $\omega_1 = 0.01, \omega_2 = 0.02, \omega_3 = 0.03, \omega_4 = 0.04$ and varying $\omega_5 = 0.001 \sim 1.0$ in the same manner, synchronization can also be achieved for all values of ω_5 for $\gamma \geq \gamma_c \approx 2.5$.

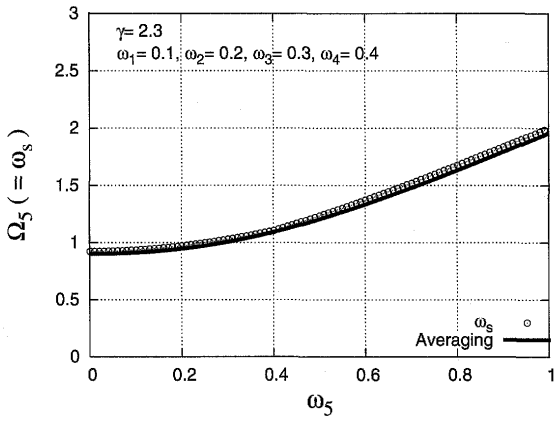
Figure 2 (a) presents the synchronized angular frequency Ω_5 in terms of ω_5. We compare this result with the empirical formula obtained in [9]: $\omega_s = \sqrt{\omega_1^2 + \gamma \sum_{l=1}^{N} \omega_l^2}$, where ω_s is the angular frequency of the same-phase synchronization and ω_1 is the maximum natural angular frequency, i.e. $\omega_1 \geq \omega_i$ for $2 \leq i \leq N$ in his notation[††]. Figure 2(b) compares the ω_5 versus ω_s characteristics from the formula with the ω_5 versus Ω_5 characteristics from our computer calculation for $\gamma = 2.3$ for example. They show

[†]It is enough to vary $0 < \omega_5 \leq 1$, because if $\omega_j/\omega_i \geq 1$, then $\omega_i/\omega_j \leq 1$.

[††]Since all components of **B** are positive, it is said from a theorem in p.171 in [13] that the largest eigenvalue of **B** (Ω_N^2) satisfies the following relation: $\Omega_N^2 \leq \omega_s^2$. Therefore, the curve from the formula is close to but larger than our simulation curve in Fig. 2(b).



(a)



(b) Comparison of Ω_5 with ω_s.

Fig. 2 Synchronized angular frequency Ω_5 in terms of ω_5 (varying from 0.001 to 1.0) for $\omega_1 = 0.1, \omega_2 = 0.2, \omega_3 = 0.3, \omega_4 = 0.4$ (a) Computer calculation result for various values of coupling factor γ (b) Comparison between our computer calculation result and Nagano's heuristic formula result for $\gamma = 2.3$.

Table 1 Comparison between computer simulation results and averaging method results for N=5, $\gamma = 1.4$, $\omega_1 = 0.1$, $\omega_2 = 0.2$, $\omega_3 = 0.3$, $\omega_4 = 0.4$ and $\omega_5 = 0.5$.

	Averaging method	Computer simulation
Ω_5	0.9772	0.9775
Amplitude of x_1	0.0164	0.0175
Amplitude of x_2	0.0679	0.0677
Amplitude of x_3	0.1615	0.1615
Amplitude of x_4	0.3124	0.3126
Amplitude of x_5	0.5505	0.5500

in good agreement. We confirmed the empirical formula agrees well for other values of γ. Synchronized angular frequency Ω_N is much larger than natural angular frequency of each oscillator. This is due to the manner of coupling adopted in Eq. (2).

Table 1 compares the result of averaging method with that of direct computer simulation of Eq. (2) for $\varepsilon = 0.1$, $\gamma = 1.4$, $\omega_1 = 0.1$, $\omega_2 = 0.2$, $\omega_3 = 0.3$, $\omega_4 = 0.4$ and $\omega_5 = 0.5$. They show in good agreement. Figure 3 demonstrates the result of direct computer simulation in this case. The oscillation with higher natural frequency suppresses the amplitude of oscillation with lower natural frequency for achieving synchronization. From the view point of dynamics of the slime, this may be reasonable as follow. An oscillator with low natural frequency corresponds to inactive slime cell and an oscillator with high natural frequency corresponds to active slime cell. When they are coupled, the contribution of the active cell becomes larger than that of the inactive cell.

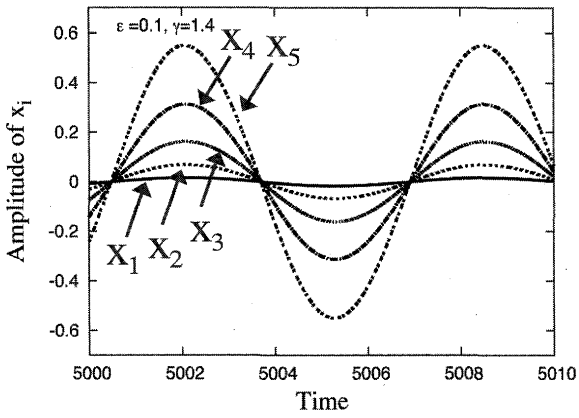


Fig. 3 Time waveforms from direct computer simulation for five coupled oscillator case for $\varepsilon = 0.1$, $\gamma = 1.4$, $\omega_1 = 0.1$, $\omega_2 = 0.2$, $\omega_3 = 0.3$, $\omega_4 = 0.4$ and $\omega_5 = 0.5$.

4.2 Synchronization Characteristics for a Large Number of Coupled Oscillators with Randomly Distributed Natural Frequencies

From the above results and our previous work [10], [14], it seems that the synchronization range becomes drastically large beyond a certain coupling factor γ_c. In particular, for two coupled van der Pol oscillator case, we have confirmed analytically that the synchronization range of the same-phase solution becomes infinitely large for $\gamma \geq 1 + \sqrt{2}$ [10]. However, for a large number of coupled oscillator cases, there are many combinations of the natural angular frequencies. Therefore, we will set natural angular frequencies according to the following statistical rule.

$$\begin{aligned} &\omega_1 = 1.0 \\ &\omega_k = r + (1 - r)\sigma_k, (2 \leq k \leq N - 1) \\ &\omega_N = r \end{aligned} \tag{13}$$

In Eq. (13), σ_k denotes the uniform random number distributing from 0 to 1.0 , and r ($0 < r < 1.0$) defines the minimum value of natural angular frequency. Namely, natural angular frequencies ω_1 and ω_N are set to 1.0 and r, respectively, and other ω_k is set uniformly random between r and 1.0.

We will show the results based on averaging method for the 10 coupled oscillator case. Here, we take 100 samples of natural angular frequencies from Eq. (13) for fixed γ and judge the stability from Eq. (11)† . Then, we obtain the synchronization probability p/q where q is the trial number and p is the number of stable samples. Figure 4 shows the synchronization probability in terms of coupling factor γ for

†We check the non-resonant condition (9) to pick up these samples.

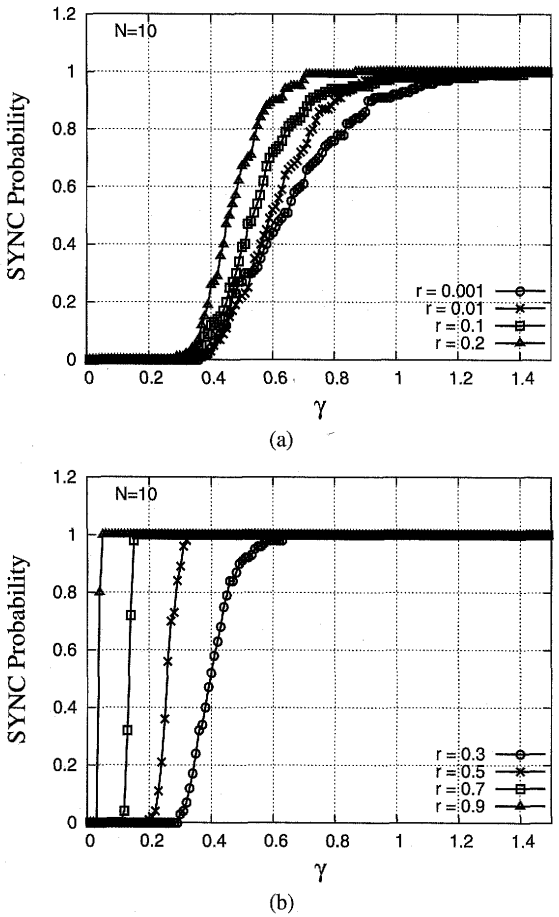


Fig. 4 Synchronization probability in terms of γ among 10 coupled oscillator case for various values of r.

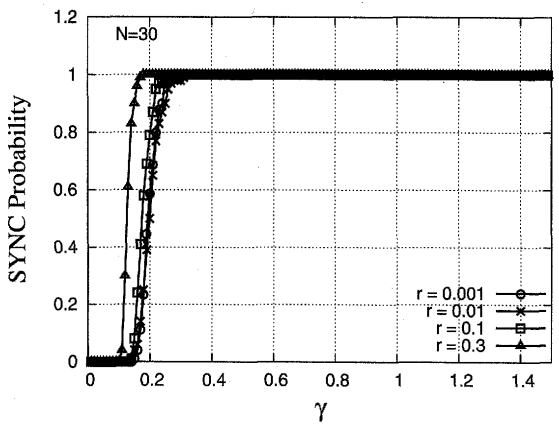


Fig. 5 Synchronization probability in terms of γ among 30 coupled oscillator case for various values of r.

various values of r. For $r \approx 0.9$, the-same phase synchronization is achieved easily. On the other hand, for $r \approx 0$, it is difficult to synchronize. However, when coupling factor γ is sufficiently large, the same-phase synchronization seems to be achieved for any r. There exist a critical γ for each r = constant curve above which synchronization is achieved for any combination of natural angular frequencies. This critical value decreases with the increase of r.

Next we will show the results for larger size of coupled oscillator case, namely $N = 30$. Figure 5 shows the

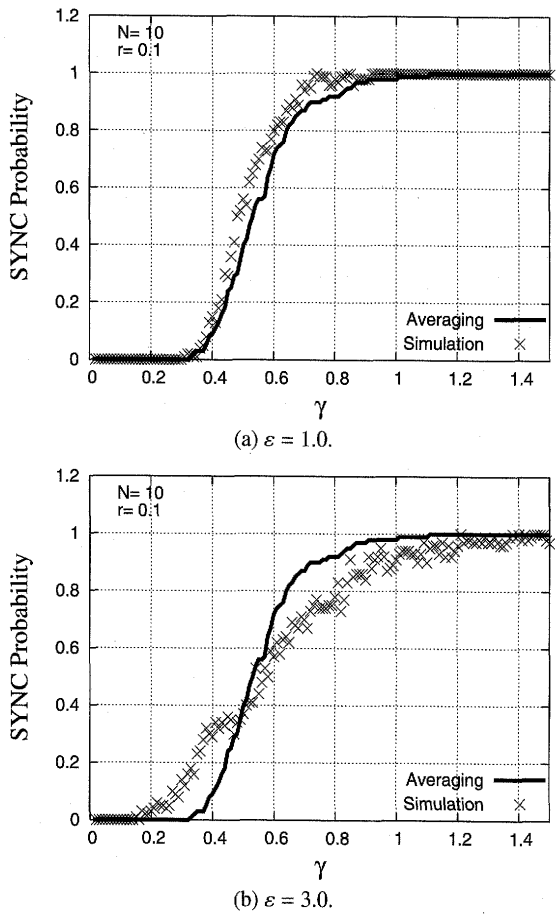


Fig. 6 Comparison of synchronization probability in terms of γ obtained by averaging method with that obtained by direct computer simulation for (a) $\varepsilon = 1.0$ and for (b) $\varepsilon = 3.0$.

synchronization probability for $r = 0.001, 0.01, 0.1$ and 0.3. It should be noted that the synchronization is achieved for comparatively small coupling factor γ compared with the 10 coupled oscillator case. This is because the natural angular frequencies are more densely distributed in limited frequency range. Therefore, a larger number of coupled oscillator systems may be more easily synchronized, though the amplitude of each oscillator is more suppressed. It seems that for $r \approx 0$ the synchronization probability curve converges to a limiting curve.

Figures 6(a) and (b) compare the synchronization probability in terms of γ obtained by averaging method with that obtained by direct computer simulation for $\varepsilon = 1.0$ and 3.0, respectively†. The computer simulation results for $\varepsilon = 1.0$ agrees well with the results of averaging method. This means that although averaging method results is guaranteed for small ε only, in practice, it is applicable for non-weak nonlinear cases around $\varepsilon = 1.0$. However, when ε becomes 3.0 these two results differ considerably.

Figures 7(a) and (b) present the coupling factor γ ver-

†In our computer simulation, many randomly chosen initial values are given to Eq. (2) and we compute the probability for the steady-state solution converging to the same-phase solution. For $\varepsilon \geq 1.0$, it seems that other modes except the same-phase become unstable.

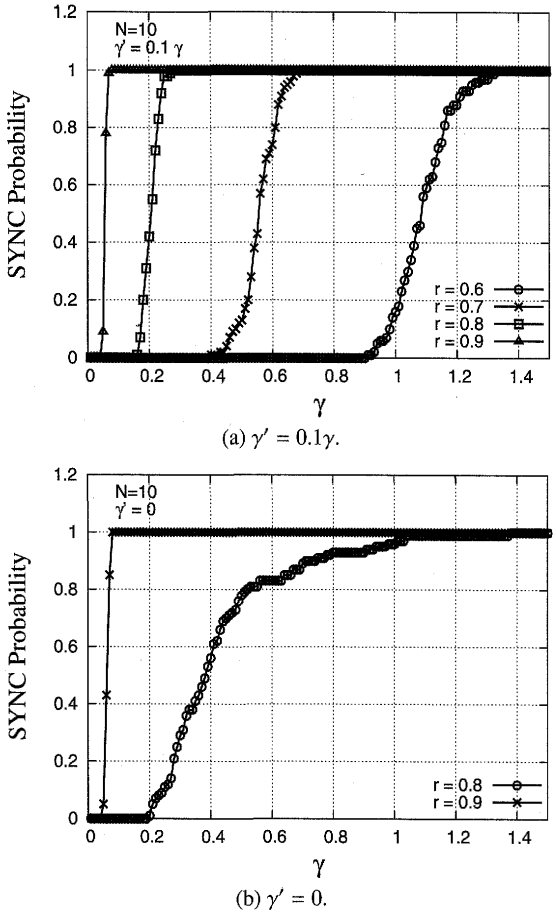


Fig. 7 Synchronization probability in terms of γ among 10 coupled oscillator case for various values of r for (a) $\gamma' = 0.1\ \gamma$; (b) $\gamma' = 0$ (linear coupling case).

sus synchronization probability for (a) $\gamma' = 0.1\gamma$ and (b) $\gamma' = 0$. In these figures, we investigate the effect of nonlinear coupling factor $\gamma' \equiv \gamma'_1 = \gamma'_2 = \cdots = \gamma'_N$. When $\gamma' = 0$, synchronization can not be achieved below $r = 0.8$. However, for $\gamma' = 0.1\gamma$, it is achieved, at least, above $r = 0.6$. Further for $\gamma' = \gamma$, it is achieved above $r \approx 0$ (in Fig. 4). From these results, the origin of strong synchronization of this coupling scheme exists in the nonlinear coupling term for such a large number of coupled system.

5. Conclusions

The averaged equation for an arbitrary number of oscillators coupled by nonlinear coupling scheme invented by S. Nagano, is derived. By using the averaged equation, we investigate the synchronization characteristics of five coupled oscillators and a large number of coupled oscillators. In particular, we present the statistical property of coupled oscillators in terms of coupling factor γ. As a result, it is confirmed that there is a critical coupling factor γ_c above which synchronization can be achieved for any combination of natural angular frequencies for a fixed r which determines the range of natural angular frequency distribution. We also investigate the origin of this strong synchronization scheme. To do this, we reduce the effect of nonlinear coupling factor as $\gamma' = 0.1\gamma$ and $\gamma' = 0$, and find that these two cases show qualitatively different synchronization characteristics. Namely, the $\gamma' = 0.1\gamma$ case can achieve synchronization for fairly small values of r. In contrast, the $\gamma' = 0$ case can achieve synchronization for $r \geq 0.8$ only. Namely, synchronization ability becomes stronger by adding even small nonlinear coupling term. As a future problem, we will investigate real slime dynamics by using this equation.

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Appendix A: Van der Pol Equation

Assuming the cubic nonlinearity for NC, i.e., $i_{NC} = -g_1 v + g_3 v^3, g_1, g_3 > 0$ in Fig. A· 1, Kirchhoff's current law gives following equation:

$$\frac{1}{L}\int v dt + C\frac{dv}{dt} - g_1 v + g_3 v^3 = 0 \qquad \text{(A·1)}$$

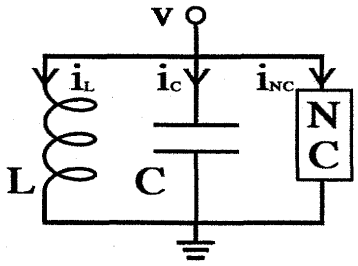


Fig. A·1 A van der Pol electric circuit where L is the inductance, C is the capacitance, and NC is the nonlinear conductance.

Taking the time derivative of Eq. (A·1), we obtain the following equation.

$$\frac{d^2v}{dt^2} - \frac{g_1}{C}(1 - \frac{3g_3}{g_1}v^2)\frac{dv}{dt} + \frac{1}{LC}v = 0 \tag{A·2}$$

By changing the variables as:

$$v = v_0 x,\ v_0 = \sqrt{\frac{g_1}{3g_3}},\ \frac{1}{\sqrt{CL}}t = \omega t'$$
$$\varepsilon = g_1\sqrt{\frac{L}{C}} \tag{A·3}$$

van der Pol equation can be derived as Eq. (1).

Appendix B: Derivation of the Function h_k in Eq. (6b)

$$\begin{aligned} h_k(\mathbf{y},\dot{\mathbf{y}}) &= \sum_{s=1}^{N} q_{ks} g_s(\mathbf{x},\dot{\mathbf{x}})\Big|_{\substack{\mathbf{x}=\mathbf{P}\mathbf{y}\\ \dot{\mathbf{x}}=\mathbf{P}\dot{\mathbf{y}}}} \\ &= \sum_{s=1}^{N} q_{ks}\omega_s\left(x_s + \gamma'_s\sum_{l=1}^{N} x_l\right)^2 \dot{x}_s\Big|_{\substack{\mathbf{x}=\mathbf{P}\mathbf{y}\\ \dot{\mathbf{x}}=\mathbf{P}\dot{\mathbf{y}}}} \end{aligned} \tag{A·4}$$

Next,

$$\begin{aligned} x_s + \gamma'_s\sum_{l=1}^{N} x_l &= \sum_{m=1}^{N} p_{sm}y_m + \gamma'_s\sum_{m=1}^{N}\left(\sum_{l=1}^{N} p_{lm}\right)y_m \\ &= \sum_{m=1}^{N}\left(p_{sm} + \gamma'_s\sum_{l=1}^{N} p_{lm}\right)y_m \\ &= \sum_{m=1}^{N} a_{sm}y_m \end{aligned} \tag{A·5}$$

where a_{sm} is defined in Eq.(6c).
Further,

$$\dot{x}_s = \sum_{m=1}^{N} p_{sm}\dot{y}_m \tag{A·6}$$

Combining Eq. (A·4)–(A·6), the function $h_k(\mathbf{y},\dot{\mathbf{y}})$ becomes as follows:

$$\begin{aligned} h_k(\mathbf{y},\dot{\mathbf{y}}) &= \sum_{s=1}^{N} q_{ks}\left(\sum_{m=1}^{N} a_{sm}y_m\right)^2\left(\sum_{m=1}^{N} p_{sm}\dot{y}_m\right)\omega_s \\ &= \sum_{s=1}^{N} q_{ks}(a_{s1}y_1 + a_{s2}y_2 + \cdots + a_{sN}y_N)^2(p_{s1}\dot{y}_1 \\ &\quad + p_{s2}\dot{y}_2 + \cdots + p_{sN}\dot{y}_N)\omega_s \\ &= \sum_{s=1}^{N} q_{ks}\omega_s\left(\sum_{m_1=1}^{N}\sum_{m_2=1}^{N}\sum_{m_3=1}^{N} a_{sm_1}a_{sm_2}p_{sm_3}y_{m_1}y_{m_2}\dot{y}_{m_3}\right) \\ &= \sum_{m_1=1}^{N}\sum_{m_2=1}^{N}\sum_{m_3=1}^{N} b(k,m_1,m_2,m_3)y_{m_1}y_{m_2}\dot{y}_{m_3} \end{aligned} \tag{A·7}$$

where $b(k,m_1,m_2,m_3)$ is defined in Eq. (6c).

Appendix C: Derivation of the Averaged Equation (8a)

Defining $\psi_k = \Omega_k t + \theta_k$, the averaged equation for $\dot{\rho}_k$ can be written as:

$$\begin{aligned} \dot{\rho}_k &= \frac{\varepsilon}{\Omega_k}\langle f_k(y,\dot{y})\cos\rho_k\rangle \\ &= \frac{\varepsilon}{\Omega_k}\left(\sum_{l=1}^{N}\tilde{c}_{kl}\rho_l\Omega_l\langle\cos\psi_l\cdot\cos\psi_k\rangle\right. \\ &\quad \left. -\sum_{m_1=1}^{N}\sum_{m_2=1}^{N}\sum_{m_3=1}^{N} b(k,m_1,m_2,m_3)\rho_{m_1}\rho_{m_2}\rho_{m_3}\Omega_{m_3}\cdot AV_1\right) \end{aligned} \tag{A·8a}$$

where

$$AV_1 \equiv \langle\sin\psi_{m_1}\cdot\sin\psi_{m_2}\cdot\cos\psi_{m_3}\cdot\cos\psi_k\rangle. \tag{A·8b}$$

The term $\langle\cos\psi_l\cdot\cos\psi_k\rangle$ becomes $1/2$ for $l = k$, because $\langle\cos^2\psi_k\rangle = \langle 1/2 + 1/2\cos 2\psi_k\rangle = 1/2$, otherwise it is zero for all $l \neq k$. Next, we will calculate AV_1. Although there are many cases, the non-zero cases are limited to the following two cases, if the non-resonant condition Eq. (9) is satisfied.

$$\begin{aligned} &\text{case 1}: m_1 = m_2 \neq k, k = m_3: \\ &AV_1 = \langle\sin^2\psi_{m_1}\cdot\cos^2\psi_k\rangle \\ &= \left\langle\frac{1}{4} - \frac{1}{4}\cos 2\psi_{m_1} + \frac{1}{4}\cos 2\psi_k\right. \\ &\quad \left. -\frac{1}{8}\cos(2\psi_{m_1} - 2\psi_k) - \frac{1}{8}\cos(2\psi_{m_1} + 2\psi_k)\right\rangle \\ &= \frac{1}{4} \end{aligned} \tag{A·9a}$$

$$\text{case 2}: m_1 = m_2 = m_3 = k: \quad AV_1 = \langle\sin^2\psi_k\cdot\cos^2\psi_k\rangle = \left\langle\frac{1}{8} - \frac{1}{8}\cos 4\psi_k\right\rangle = \frac{1}{8} \tag{A·9b}$$

Therefore, the averaged equation for $\dot{\rho}_k$ becomes as Eq. (7).

Next, the averaged equation for $\dot{\theta}_k$ can be written as :

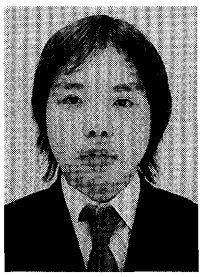
$$\dot{\theta}_k = \frac{-\varepsilon}{\Omega_k\rho_k}\langle f_k(y,\dot{y})\sin\psi_k\rangle$$

$$= \frac{-\varepsilon}{\Omega_k \rho_k} \left(\sum_{l=1}^{N} \tilde{c}_{kl} \rho_l \Omega_l \langle \cos\psi_l \cdot \sin\psi_k \rangle - \sum_{m_1=1}^{N} \sum_{m_2=1}^{N} \sum_{m_3=1}^{N} b(k, m_1, m_2, m_3) \rho_{m_1} \rho_{m_2} \rho_{m_3} \Omega_{m_3} \cdot AV_2 \right) \quad (A \cdot 10a)$$

where

$$AV_2 \equiv \langle \sin\psi_{m_1} \cdot \sin\psi_{m_2} \cdot \cos\psi_{m_3} \cdot \sin\psi_k \rangle . \quad (A \cdot 10b)$$

The term $\langle \cos\psi_l \cdot \sin\psi_k \rangle$ is equal to zero for all k and l. The term AV_2 becomes zero for all possible combinations of m_1, m_2, m_3 and k, if the non-resonant condition Eq. (9) is satisfied. Therefore, the averaged equation for $\dot{\theta}_k$ becomes as Eq. (7).



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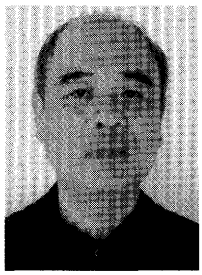

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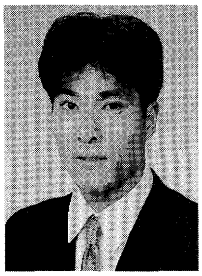

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