Optimal Waveform for the Entrainment of a Weakly Forced Oscillator

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A theory for obtaining a waveform for the effective entrainment of a weakly forced oscillator is presented. Phase model analysis is combined with calculus of variation to derive a waveform with which entrainment of an oscillator is achieved with a minimum power forcing signal. Optimal waveforms are calculated from the phase response curve and a solution to a balancing condition. The theory is tested in chemical entrainment experiments in which oscillations close to and farther away from a Hopf bifurcation exhibited sinusoidal and higher harmonic nontrivial optimal waveforms, respectively.

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Entrainment of oscillators to an external signal in nonlinear dissipative systems is a fundamental concept of importance in a large variety of applications [1]. Two prominent examples include the time-scale adjustment of the circadian system to light [2] and the cardiac system to a pacemaker [3]. The entrainment process is rigorously described theoretically by phase or amplitude equations and circle maps for weakly and strongly perturbed nonlinear systems, respectively [1]. The general result of the theoretical analysis is that nonlinear oscillators can adjust their frequencies to that of the external source above a critical forcing amplitude. In the forcing amplitude versus forcing frequency diagram there are long vertical entrainment regions called Arnold tongues.

A widely accepted tool for studying entrainment is the phase response curve (PRC) [2]. Specifically, the phase response function (infinitesimal PRC) indicates the phase shift of an oscillator due to an infinitesimal perturbation of a system variable [4]. A classical problem in nonlinear dynamics uses the phase response function and forcing waveform with which all important features of the entrainment process (e.g., locking range, defined as the width of the Arnold tongue at a given forcing amplitude) can be obtained for weakly perturbed systems [1].

Many applications require optimization of the entrainment process. This is often achieved by adjusting the forcing waveform to achieve a target entrainment feature. A variety of control targets were explored: optimal input was determined for establishing fast entrainments [5], circadian phase resetting [6,7], starting or stopping of the oscillations [7,8], and maximal resonance (energy transfer) between the system and the forcing signal [9]. Control of deterministic [10] and stochastic [11,12] neuronal spiking activity was achieved with a phase modeling approach combined with variational methods to optimize spiking time [10] and variance of firing rates [12].

In this Letter, we give a full account for the inverse of the classical entrainment problem: namely, What is the minimal power forcing waveform that produces efficient entrainment of a limit-cycle oscillator in the weak forcing limitation? Although the quality of entrainment could involve features such as stability and basin of attraction, here we consider efficient entrainment as the occurrence of maximum width (or minimum slopes) of the Arnold tongue. This ‘‘locking range’’ quality marker has been commonly used for injection-locked microintegrated oscillators as well as phase-locked loop circuits [13]. We propose a versatile, efficient approach to obtain an exact functional form of the optimal waveform provided that the response function related to the forcing action had been established. The theoretically obtained optimal waveforms, which exhibit some unexpected symmetry relationships with the response function, are tested in a simple numerical model that include higher harmonics in the response function typically seen in strongly nonlinear oscillators. The experimental applicability of the method is demonstrated with optimal entrainment of a chemical system, the oscillatory electrodissolution of nickel in sulfuric acid.

The entrainment process of a limit-cycle oscillator in the weak forcing limit can be modeled by [14]

$$\frac{d\psi}{dt} = \omega + Z(\psi)f(\Omega t), \quad (1)$$

where ψ is the phase of the oscillator, Z is the phase response function, and ω and Ω are the natural frequency of the oscillator and the forcing frequency, respectively. In the weak forcing limit, Eq. (1) is further simplified by averaging [15], as

$$\frac{d\phi}{dt} = \Delta\omega + \Gamma(\phi), \quad (2)$$

where ϕ and $\Delta\omega$ are given by $\phi = \psi - \Omega t$ and $\Delta\omega = \Omega - \omega$, respectively [16]. The interaction function $\Gamma(\phi)$ is obtained from the forcing waveform f and the phase response function Z, as $\Gamma(\phi) = \langle Z(\theta + \phi)f(\theta)\rangle$, where θ represents Ωt and $\langle\cdot\rangle$ denotes the average by θ over its period 2π: $\langle\cdot\rangle \equiv (2\pi)^{-1}\oint d\theta$. Entrainment occurs when the phase difference is locked, i.e., $d\phi/dt = \Delta\omega + \Gamma(\phi) = 0$

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[7]. The range of frequency difference $\Delta\omega$, where the solution for stable steady state exists for ϕ, defines the locking range $R[f]$ for a certain forcing waveform [17]. Therefore, the locking range is the difference between the maximum (at $\phi = \phi_+$) and minimum (at $\phi = \phi_-$) values of $\Gamma(\phi)$ where a phase-locked solution exists [18]. $R[f]$ is thus given by $\Gamma(\phi_+) - \Gamma(\phi_-)$.

Now we are in position to formulate the optimal entrainment problem mathematically: the optimal forcing waveform ($f_{\rm opt}$) maximizes the locking range R under certain constraints. A convenient practical constraint is the total power of the waveform over its period, $\langle f(\theta)^2\rangle$. Therefore, the optimal forcing waveform $f_{\rm opt}$ gives maximal locking range for a given (constant) forcing power P. We consider this as a variational problem maximizing the functional form

$$\mathcal{S}[f] \equiv R[f] - \lambda(\langle f^2\rangle - P), \tag{3}$$

where λ is the Lagrange multiplier. The solution to the variational problem f_*, a suitable candidate for the optimal waveform, is obtained by ensuring that the first variation $\delta\mathcal{S}$ vanishes and the second variation $\delta^2\mathcal{S}$ is negative [19]:

$$f_*(\theta) = (2\lambda)^{-1}\{Z(\theta + \phi_+) - Z(\theta + \phi_-)\}. \tag{4}$$

The Lagrange multiplier can be obtained by substituting the solution of Eq. (4) in the constant power constraint $(\langle f_*^2\rangle - P = 0)$: $\lambda = (1/2)\sqrt{Q/P}$ with $Q \equiv \langle\{Z(\theta + \phi_+) - Z(\theta + \phi_-)\}^2\rangle$. Note that f_* in Eq. (4) has zero average: $\langle f_*\rangle = 0$.

To obtain f_* of Eq. (4) the maximum (ϕ_+) and the minimum (ϕ_-) ϕ values of Γ with the forcing waveform,

$$\Gamma(\phi) = \sqrt{P/Q}\langle Z(\theta + \phi)\{Z(\theta + \phi_+) - Z(\theta + \phi_-)\}\rangle, \tag{5}$$

have to be determined. The conditions for the maximum and minimum of Γ are as follows:

$$\Gamma'(\phi_\pm) = 0, \qquad \Gamma''(\phi_+) < 0, \quad \text{and} \quad \Gamma''(\phi_-) > 0. \tag{6}$$

The first condition in Eq. (6), combined with Eq. (5), gives

$$\langle Z'(\theta + \phi_+)Z(\theta + \phi_-)\rangle = \langle Z'(\theta + \Delta\phi)Z(\theta)\rangle = 0, \tag{7}$$

where $\Delta\phi \equiv \phi_+ - \phi_-$. ($\Delta\phi$ is introduced to remove phase ambiguity.) We shall refer to Eq. (7) as the balancing condition because this equation realizes optimality by balancing both terms in Eq. (4). The trivial $\Delta\phi = 0$ solution to Eq. (7) is discarded because it does not allow entrainment [$\Gamma(\phi) \equiv 0$ in Eq. (5)].

However, other solutions do exist in Eq. (7), because $\partial\langle Z'(\theta + \Delta\phi)Z(\theta)\rangle/\partial\Delta\phi|_{\Delta\phi=0} = -\langle Z'(\theta)^2\rangle < 0$, and $\langle Z'(\theta + \Delta\phi)Z(\theta)\rangle$ is a periodic, bounded function of $\Delta\phi$ in a large class of systems [1]. In particular, we have found that in models with twice differentiable, continuous Z a solution with $\Delta\phi = \pi$ exists; we call this solution and the corresponding optimal waveform "generic" [20].

Here we test the above theoretical predictions by a simple model with the following response function:

$$Z(\theta) = \sin\theta + a\sin(2\theta). \tag{8}$$

This Z simulates the behavior of the Stuart-Landau oscillator [1] with $a = 0$; therefore, we can consider a as a measure of distance from Hopf bifurcation that can introduce higher harmonics in the response function. The balancing condition of Eq. (7) for this model is explicitly written as $[1 + 4a^2\cos(\Delta\phi)]\sin(\Delta\phi) = 0$. For $|a| < 1/2$, there is only one (nontrivial) solution: $\Delta\phi = \pi$. The optimal waveform for this generic solution is independent of a: $f_{\rm opt}(\theta) = -\sqrt{2P}\sin\theta$. Although the response function does contain a second order harmonic for $0 < |a| < 1/2$, this term does not affect the shape of the optimal waveform. This finding suggests that with Z containing only weak second (or, in general, even) harmonics, the sinusoidal forcing is the optimal since the generic solution to the balancing condition always exists. For example, for systems close to Hopf bifurcation, which contain mostly first and weak second (even) harmonics in Z, the optimal waveform is sinusoidal. However, the odd harmonics do appear in the generic optimal waveform, and thus systems with relatively strong third (and higher odd) harmonics in Z are expected to retain the odd harmonics in the optimal waveform. For the generic solution the locking range is calculated as $R = \sqrt{2P}$, which is again independent of a.

In the range $|a| \geq 1/2$, we have three different solutions for the balancing condition and thus three candidates for the optimal waveform. The generic solution with $\Delta\phi = \pi$ exists; however, there appear two additional, "nongeneric" solutions satisfying $\cos\Delta\phi = -1/4a^2$. For these nongeneric solutions the locking ranges are identical: $R = (1 + 4a^2)\sqrt{P}/(2\sqrt{2}a) \geq \sqrt{2P}$. This shows that in the range $|a| \geq 1/2$, the nongeneric optimal waveforms outperform the generic waveform; for large values of a, the improvement of locking range for the nongeneric over generic waveforms increases approximately linearly. The nongeneric waveforms are not purely sinusoidal and depend on the parameter a. Two nongeneric waveforms for $a = 0.95$ are shown in Fig. 1(a).

We have also verified these theoretical predictions, by using a standard genetic algorithm which numerically searches for $f_{\rm opt}$. Figure 1(a) shows all $f_{\rm opt}$ obtained by the algorithm, for both cases of full [Eq. (1)] and averaged [Eq. (2)] phase models, with $\omega = 10$, $a = 0.95$ ($>1/2$) and $P = 0.5$ or $P = 10.0$. The numerical algorithm found the exact same optimal waveform as predicted by the theory, for both cases up to around $P = 2.0$. The numerical value of the locking range for these optimal solutions [maximum in R landscape in Fig. 1(b)] is also the same as predicted by the theory; the nongeneric optimal solution performs about 23.1% better than the simple sinusoidal (generic optimal) forcing, and it also performs about 89.8% better than the pulse forcing in Fig. 2(c). Beyond $P = 2.0$, a small discrepancy appears in the optimal waveforms obtained with the genetic algorithm. However, the shape of the bimodal landscapes of $R[f]$ in Fig. 1(b) is preserved up to around $P = 10.0$ for the case of Eq. (1), which

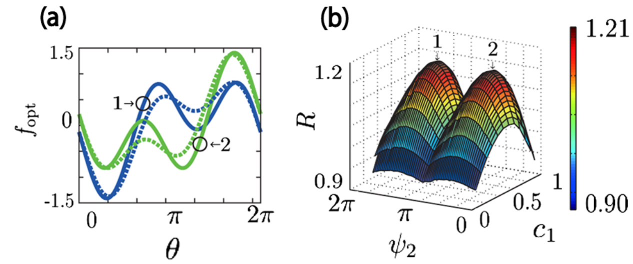

FIG. 1 (color online). Optimal forcing $f_{\rm opt}$ obtained in the model with the response function of Eq. (8). Panel (a) shows all $f_{\rm opt}$, after rescaling for comparison, obtained by the genetic algorithm for $a = 0.95$: solid curves for $P = 0.5$ with Eq. (2), dotted curves for $P = 10.0$ with Eq. (1). In this genetic algorithm, $f_{\rm opt}$ is searched among all functions of the form $c_1 \sin\theta + c_2 \sin(2\theta + \psi_2)$. Panel (b) shows the landscape of R for $a = 0.95$ ($>1/2$), with respect to (c_1, ψ_2). The peaks 1 and 2 of the landscape correspond to the waveforms 1 and 2 in (a), respectively.

suggests that, at least in this particular example, beyond the strictly weak forcing limit the theoretical prediction of optimal waveform could be used as an initial candidate that can be further optimized with other techniques.

The theoretical method for obtaining the optimal forcing waveform was demonstrated in chemical experiments with Ni electrodissolution. The experiments were carried out in a standard three-electrode electrochemical apparatus with a 1 mm diameter Ni wire as working, a 1.57 mm diameter Pt coated Ti rod as the counter, and $Hg/Hg_2SO_4/K_2SO_4$(sat) as the reference electrode. The potential of the Ni wire was set to a circuit potential $V = V_0 + Af_0(\theta)$, where V_0 is a base potential, A is the amplitude of forcing, and $f_0(\theta)$ is the normalized forcing waveform with $\langle f_0^2 \rangle = 0.5$. The current, proportional to dissolution rate, was measured by the potentiostat. The frequency of the current oscillations was determined with the Hilbert transform method [1]. The experiments were carried out at 10 °C in 3 mol/L sulfuric acid solution. Further experimental details are given in previous studies [21,22]. When a resistor of 1 kΩ is attached to the Ni electrode, oscillations arise through Hopf bifurcation at $V_0 \approx 1.07$ V due to the hidden negative differential resistance of the electrochemical system [23].

In accordance with previous experimental results [22], at $V_0 = 1.100$ V, slightly above the bifurcation point, the oscillations are smooth with a PRC that contains predominantly first harmonic components [see Figs. 2(a) and 2(b)]. The experiments were carried out at fixed forcing frequencies and the amplitude of the waveform was increased until entrainment occurs where the critical amplitude A_c was recorded. To compare the forcing capability of different waveforms we determined the reduced entrainabilities as $E_r = |\Delta\omega|/(A_c\sqrt{\langle f_0^2\rangle})$. The average value of the reduced entrainability for fixed positive and negative detunings is proportional to the R value; therefore, it is a useful quantity to characterize the entraining capability of a waveform. In

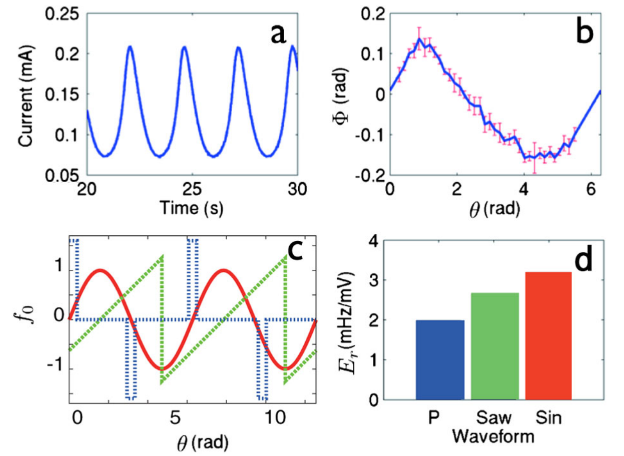

FIG. 2 (color online). Experiments: Optimal entrainment of smooth oscillators, $V_0 = 1.100$ V. (a) Waveform. (b) Phase response curve (PRC). The PRC was obtained with pulsing the oscillations with an amplitude of −100 mV and width of 0.1 rad and measuring the phase advance (Φ) in five independent experiments. The error bars indicate standard deviations in the five experiments. (c) Three waveforms, sine-sawtooth-pulse, with which entrainments are tested. (d) Reduced entrainabilities of the three waveforms: P, pulse; Saw, sawtooth; Sin, sine. The detuning $\Delta\omega/\omega$ was set to ±5%; at each detuning two experiments were carried out.

a typical set of experiments 5% detuning was applied, i.e., $\Delta\omega/\omega = \pm 0.05$; it was confirmed that smaller detuning (2%) gives E_r values within 5% relative error. Experiments with larger detunings resulted in large changes of oscillation waveforms and thus the weak forcing assumption of the theory was not satisfied.

We have tested three waveforms with smooth oscillators; sine, sawtooth, and pulse waves are shown in Fig. 2(c). The optimal waveform was determined by the above algorithm [24]; only the generic waveform exists, and because the PRC has very small third and higher harmonics (less than 1%), we can consider the sine waveform as optimal within experimental error. The entrainabilities in Fig. 2(d) show that, as predicted, the sine waveform indeed outperforms the sawtooth and pulse waves.

When V_0 is increased to 1200 mV, the oscillations become moderately relaxational; the waveform and the PRC exhibit higher harmonics [see Figs. 3(a) and 3(b)]. The analysis has revealed that three optimal waveforms exist: one generic and two nongeneric, given in Figs. 3(c)–3(e), respectively. In Fig. 3(e) we also show a waveform that is obtained by adding Gaussian random numbers with standard deviation of 0.1 to the Fourier coefficients of the most optimal waveform; this waveform is referred to as noisy optimal 2 waveform. The reduced entrainabilities follow the theoretical predictions: sawtooth, sine, and pulse waves are inferior to the entraining power of the optimal waveforms. As predicted, the generic optimal waveform performs slightly worse than the nongeneric optimal waveforms. The best waveform was optimal 2;

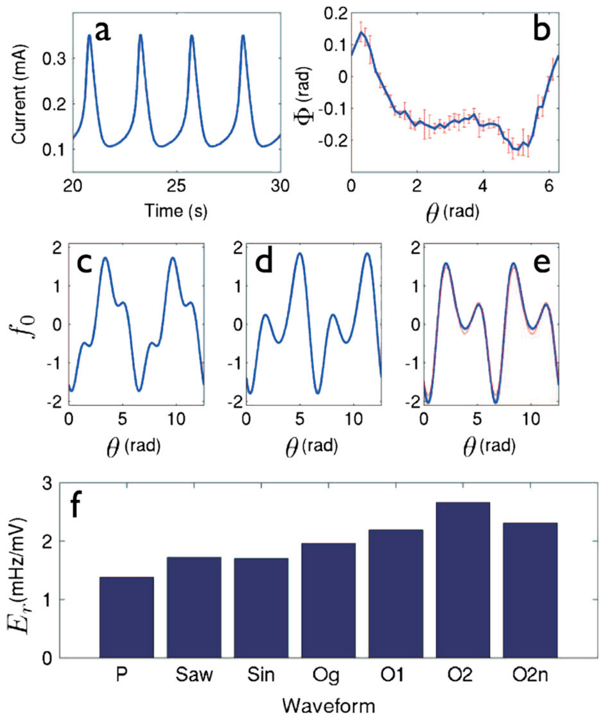

FIG. 3 (color online). Experiments: Optimal entrainment of moderately relaxational oscillators, $V_0 = 1.200$ V. (a) Oscillation waveform. (b) Phase response curve. The PRC was obtained similarly to that in Fig. 2. (c)–(e) Optimal waveforms: (c) Generic optimal waveform (Og). (d) Optimal wave form 1 (O1). (e) Thick solid line: optimal waveform 2 (O2). Thin line: noisy optimal waveform 2 (O2n). (f) Reduced entrainabilities of the tested waveforms. The E_r values are averages of experiments with ±2% and ±5% detunings.

note that the shape of this waveform is strongly nontrivial and by the addition of a small amount of noise to the waveform the E_r value decreases. Overall, the optimal waveform obtained from the theory increased entrainabilities by about 50% to 90% relative to those obtained with sine-pulse-sawtooth waves.

Construction of an optimal waveform for entrainment was proposed and tested here with a single oscillator. The method, however, can be extended to a group of interacting oscillators where effects related to the collective phase response function [25] shall be considered. The optimal signal can also be applied in closed-loop feedback systems along with synchronization engineering [21] for seeking optimal target dynamics. A limitation of the methodology is the requirement for weak forcing so that phase models can be applied. This limitation leads to an extension of the method where the forcing signal is limited to small values. These extensions, along with other targets that consider stability and basin of attraction, will be considered in a forthcoming publication. The proposed methodology provides a framework for efficient design of entrainment applications in electrical circuit technology (e.g., for injection-locked oscillators) as well as in biological pacemakers.

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