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Melnikov Analysis for a Second Order Phase-Locked Loop in the Presence of a Weak CW Interference

Hisa-Aki TANAKA†, Shin'ichi OISHI† ***and*** **Kazuo HORIUCHI†,** ***Members***

SUMMARY This letter presents the results of an analysis concerning the global, dynamical structure of a second order phase-locked loop (PLL) in the presence of the continuous wave (CW) interference. The invariant manifolds of the PLL equation are focused and analyzed as to how they are extended from the hyperbolic periodic orbits. Using the Melnikov integral which evaluates the distance between the stable manifolds and the unstable manifolds, the transversal intersection of these manifolds is proven to occur under some conditions on the power of the interference and the angular frequency difference between the signal and the interference. Numerical computations were performed to confirm the transversal intersection of the system-generated invariant manifolds for a practical set of parameters.

key words: *phase-locked loop, CW interference, Melnikov analysis*

1. Introduction

In today's coherent communication systems, phase-locked loops (PLL's) are used extensively for demodulation of the signal. In such systems, many users always coexist in the same frequency channel for communication. Consequently, performance of PLL's in the presence of the continuous wave (CW) interference is a problem of practical interest, and has been studied by many authors.

From the viewpoint of nonlinear dynamical systems, this problem has been investigated by Endo and Iizuka [1], Endo, Matsubara, and Ohta [2], Endo and Suzuki [3], recently by Oie, Iritani, and Kawakami [4], and Takahashi [5]. The earlier work by Endo and Iizuka [1] clarified the tracking performance of PLL by studying the periodic solution via the harmonic balance method. Moreover, they showed that in the initial phase plane, a complex basin boundary is formed between the basin of a periodic solution for the signal and the basin of another periodic solution for the interference, when the interference is sufficiently large. Adding to this phenomenon, long transients to the stable fixed point (i.e., *lock-in* of PLL) are often numerically observed under some set of parameters and initial states. The works [1]-[5] have investigated the tracking behavior to the signal or to the interference in detail. However, the mechanism behind these complex phenomena has not been fully elucidated.

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† The authors are with the School of Science and Engineering, Waseda University, Tokyo, 169 Japan.

This situation led to the present work which analyzes the mechanism of these complex behaviors by focusing the geometric structure of the invariant manifolds. To do so, we applied the Melnikov's method to the problem by utilizing the presence of small parameters. As a result of the analysis, the transversal intersections of the invariant manifolds is proven to occur under some conditions on the power of the interference and the angular frequency difference between the signal and the interference. In addition, we numerically confirmed the transversal intersection of the system-generated invariant manifolds for a practical set of parameters.

2. PLL Equation in the Presence of CW Interference

When the desired signal $s(t)$ takes the following form,

$$s(t)=S\sin\omega t, \tag{1}$$

the CW interference $I(t)$ can be described by

$$I(t)=\sum_{i=1}^{N} I_i \sin(\omega+\nu_i)t, \tag{2}$$

where ν_i denotes the angular frequency difference between the desired signal and the i-th mode of the CW interference. Hence, the received signal is the sum of $s(t)$ and $I(t)$, while the output $v(t)$ at the voltage-controlled oscillator (VCO) can be described by

$$v(t)=V\cos(\omega t+\theta_0), \tag{3}$$

where θ_0 denotes the phase error between the desired signal and the output at VCO. Considered here are PLL's that incorporate a voltage-controlled oscillator (VCO), a phase detector (PD) having sinusoidal characteristics, and a loop filter (LF) comprised of a simple RC filter with transfer function $F(S)=1/(1+\tau S)$, which is known as a lag filter.

Figure 1 shows a block diagram of the system, and the following ordinary differential equation (ODE), i.e., the *phase model* [1], describes the dynamics of the phase error θ_0:

$$\left(1+\tau\frac{d}{dt}\right)\frac{d\theta_0}{dt}=K_0\left\{g(-\theta_0)+\sum_{i=1}^{N}R_i g(\nu_i t-\theta_0)\right\}, \tag{4}$$

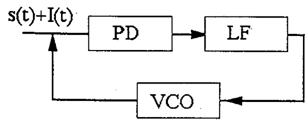

Fig. 1 Block diagram of a phase-locked loop (PLL) in the presence of a continuous wave (CW) interference.

where K_0 denotes the total loop gain, and R_i denotes the ratio I_i/S. Let the PD characteristics be sinusoidal, i.e., $g(\theta)=\sin\theta$, then the phase model is obtained by applying the trigonometric identity to (4) as

$$\ddot{\theta}_0+\tau^{-1}\dot{\theta}_0+K_0\tau^{-1}\sin\theta_0 + K_0\tau^{-1}\sum_{i=1}^{N}R_i(\cos\nu_i t\sin\theta_0-\sin\nu_i t\cos\theta_0)=0. \tag{5}$$

By a scaling of the independent variable: $t\rightarrow(\sqrt{K_0/\tau})\,t$, (5) gives

$$\ddot{\theta}_0+(1/\sqrt{K_0\tau})\,\dot{\theta}_0+\sin\theta_0 - \sum_{i=1}^{N}R_i(\cos\nu_i t\sin\theta_0-\sin\nu_i t\cos\theta_0)=0. \tag{6}$$

3. The Melnikov Analysis for PLL Equation

Denoting $\theta_0=x_1$, $\dot{\theta}_0=x_2$, and $\nu_i t=\Theta_i$, (6) is given as a first order system of ODE

$$\dot{x}_1=x_2, \tag{7a}$$

$$\dot{x}_2=-\sin x_1-(1/\sqrt{K_0\tau})\,x_2 + \sum_{i=1}^{N}R_i(\cos\Theta_i\sin x_1-\sin\Theta_i\cos x_1), \tag{7b}$$

$$\dot{\Theta}_i=\nu_i, \tag{7c}$$

where $(x_1, x_2, \Theta_i)\in(R^1\times R^1\times T^1)$ with $i=1,\cdots,N$. We concentrate here the case of $N=1$, i.e., a single tone CW interference. The general case of $N>1$ will be discussed in the near future. Since the forcing term in (7b) is periodic with respect to Θ_1, the Poincaré section can be defined on the plane Θ_1=const., where the Melnikov's method can be applied to evaluate the distance between the system-generated invariant manifolds [7]. To apply this method, the simplest case of $R_1=O(\varepsilon)$, $1/\sqrt{K_0\tau}=O(\varepsilon)$ and $\nu_1=O(1)$ is primarily considered here.

3.1 Separation of the Invariant Manifolds

In the limit of $R_1\rightarrow 0$ and $1/\sqrt{K_0\tau}\rightarrow 0$, there exist a pair of heteroclinic trajectories as in Fig. 2.

In this limit, the (7a) and (7b) component becomes Hamiltonian system with Hamiltonian $H=x_2^2/2-\cos x_1$, which gives the analytic solution on the invariant manifolds as follows,

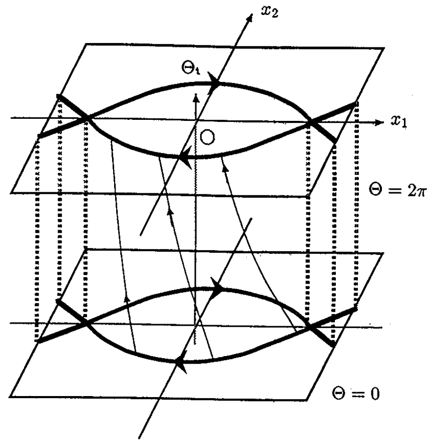

Fig. 2 Heteroclinic trajectories in the three-dimensional vector field.

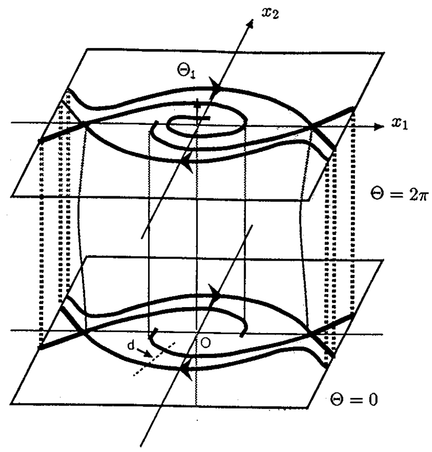

Fig. 3 Separation of the stable manifolds (W^s) and the unstable manifold (W^u).

$$x_{1h}^{\pm}=\pm 2\arcsin(\tanh t),$$

$$x_{2h}^{\pm}=\pm 2\,\mathrm{sech}\,t,$$

$$\Theta_1=\nu_1 t+\Theta_0. \tag{8}$$

In addition, the solution $(x_1, x_2, \Theta_1)=(\pm\pi, 0, \nu_1 t+\Theta_0)$ represents a hyperbolic periodic orbit respectively. When $R_1=O(\varepsilon)$ and $1/\sqrt{K_0\tau}=O(\varepsilon)$, such hyperbolic periodic orbits persist and the invariant manifolds from the hyperbolic periodic orbits generically separate into the stable manifolds (W^s) and the unstable manifolds (W^u) as in Fig. 3.

The distance d between these manifolds depicted in Fig. 3 can be evaluated via the Melnikov integral M by

$$d=\frac{M^{\pm}(t_0, \Theta_0, \nu_1, K_0\tau, R_1)}{\|D_xH(x_{1h}^{\pm}(-t_0), x_{2h}^{\pm}(-t_0))\|}+O(\varepsilon^2), \tag{9}$$

where

$$M^{\pm}(t_0, \Theta_0, \nu_1, K_0\tau, R_1) =\int_{-\infty}^{\infty}\langle D_xH, g\rangle(x_{1h}^{\pm}, x_{2h}^{\pm})\,dt, \tag{10}$$

with $D_xH=(\sin x_1, x_2, 0)$ and $g=(0, -(1/\sqrt{K_0\tau})x_2 + R_1(\cos\Theta_1 \sin x_1 - \sin\Theta_1 \cos x_1), 0)$. $\langle , \rangle$ denotes the inner product of the vectors. Since (7) is not invariant under the transformation: $(x_1, x_2)\rightarrow(-x_1, -x_2)$, $M^+ = M^-$ does not necessarily hold. $M^\pm$ can be expressed by the sum of the following integrals as

$$\begin{aligned} M^\pm(t_0, \Theta_0, \nu_1, K_0\tau, R_1) &= \int_{-\infty}^{\infty} -1/\sqrt{K_0\tau}\, x_{2h}^{\pm}(t-t_0)^2 dt \\ &+ \int_{-\infty}^{\infty} R_1 x_{2h}^{\pm}(t-t_0)\cos\Theta_1 \sin x_{1h}^{\pm}(t-t_0)\, dt \\ &- \int_{-\infty}^{\infty} R_1 x_{2h}^{\pm}(t-t_0)\sin\Theta_1 \cos x_{2h}^{\pm}(t-t_0)\, dt \\ &= I_1+I_2+I_3, \end{aligned} \tag{11}$$

in which I_1, I_2, and I_3 can be integrated analytically via the residue calculation (see [6] for a similar technique), to give the following

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} -(1/\sqrt{K_0\tau})\, x_{2h}^{\pm}(t-t_0)^2 dt \\ &= \int_{-\infty}^{\infty} -(1/\sqrt{K_0\tau})\, 4\,\mathrm{sech}^2(t-t_0)\, dt \\ &= -8/\sqrt{K_0\tau}. \end{aligned} \tag{12}$$

$$\begin{aligned} I_2 &= \int_{-\infty}^{\infty} R_1 x_{2h}^{\pm}(t-t_0)\cos\Theta_1 \sin x_{1h}^{\pm}(t-t_0)\, dt \\ &= \int_{-\infty}^{\infty} -4R_1 \frac{\sinh(t-t_0)}{\cosh^3(t-t_0)} \\ &\quad \cdot \sin\nu_1(t-t_0)\sin(\nu_1 t_0+\theta_0)\, dt \\ &= -4R_1\{\pi\nu_1^2/(2\sinh\pi\nu_1/2)\}\sin(\nu_1 t_0+\theta_0). \end{aligned} \tag{13}$$

$$\begin{aligned} I_3 &= \int_{-\infty}^{\infty} -R_1 x_{2h}^{\pm}(t-t_0)\sin\Theta_1 \cos x_{2h}^{\pm}(t-t_0)\, dt \\ &= -\int_{-\infty}^{\infty} \pm 2R_1\,\mathrm{sech}(t-t_0)\cos\nu_1(t-t_0) \\ &\quad \cdot \sin(\nu_1 t_0+\theta_0)\, dt \\ &+ \int_{-\infty}^{\infty} \pm 4R_1 \frac{\sinh^2(t-t_0)}{\cosh^3(t-t_0)} \\ &\quad \cdot \cos\nu_1(t-t_0)\sin(\nu_1 t_0+\theta_0)\, dt \\ &= \mp 2R_1\pi\sinh(\pi\nu_1/2)\sin(\nu_1 t_0+\theta_0) \\ &\quad \pm 4R_1\{\pi(1-\nu_1^2)/(2\cosh\pi\nu_1/2)\}\sin(\nu_1 t_0+\theta_0). \end{aligned} \tag{14}$$

Hence, $M^\pm$ is given by

$$\begin{aligned} M^\pm &= -\frac{8}{\sqrt{K_0\tau}} \\ &+ \pi R_1\left\{\frac{\pm 2(1-\nu_1^2)}{\cosh(\pi\nu_1/2)} - \frac{2\nu_1^2}{\sinh(\pi\nu_1/2)}\right. \\ &\left.\mp 2\sinh(\pi\nu_1/2)\right\}\sin(\nu_1 t_0+\theta_0), \end{aligned} \tag{15}$$

while

$$\begin{aligned} &\|D_xH(x_{1h}^{\pm}(-t_0), x_{2h}^{\pm}(-t_0))\| \\ &= 2\sqrt{\sinh^2(-t_0)+\cosh^2(-t_0)}/\cosh^2(-t_0) \\ &= 2\sqrt{\cosh 2t_0}/\cosh^2 t_0 > 0 \end{aligned} \tag{16}$$

holds. From (15) and (16), it suffices that we check if the Melnikov integral M has the simple zeros to determine if the transversal intersection of the stable manifolds (W^s) and the unstable manifolds (W^u) occurs. The assumption $\nu_1=O(1)$ comes from the following reasons. To evaluate the distance d between W^s and W^u via the Melnikov integral, I_1, I_2 and I_3 in M must be $O(\varepsilon)$ respectively, as in (9). However, if we consider a large value of ν_1, e.g., $\nu_1=O(1/\varepsilon)$, I_2 and the term $4R_1\{\pi(1-\nu_1^2)/(2\cosh\pi\nu_1/2)\}$ in I_3 becomes $O(e^{-1/\varepsilon})$, being smaller than any power of ε. If we consider the case of $\nu_1=O(1)$, $R_1=O(\varepsilon)$ and $1/\sqrt{K_0\tau}=O(\varepsilon)$, it is realized that this difficulty does not occur (Theoretical breakthrough on this difficulty has been made by Holmes, Marsden and Scheurle [8]).

3.2 Geometric Structure of the Invariant Manifolds

Since a second kind of periodic solution corresponding to the interference signal may appear in the upper half plane of (x_1, x_2), we focus our attention to the upper half plane here. Using the Melnikov integral given by (15), we can consider the geometric structure of the invariant manifolds W^s and W^u. It is realized that (15) is the sum of the constant term $-8/\sqrt{K_0\tau}$ and the oscillating term whose amplitude is $\pi R_1|2(1-\nu_1^2)/\cosh(\pi\nu_1/2) - 2\nu_1^2/\sinh(\pi\nu_1/2) - 2\sinh(\pi\nu_1/2)|$. Hence, the following proposition holds.

Proposition 1: For parameters satisfying $R_1=O(\varepsilon)$, $1/\sqrt{K_0\tau}=O(\varepsilon)$ and $\nu_1=O(1)$, there exists a critical parameter R_c such that the following holds:

(a) For $R_1<R_c$, $d<0$ ($\forall t_0, \theta_0$), i.e., W^s and W^u separate.

(b) At $R_1=R_c$, $d=0$ ($\exists t_0, \theta_0$), i.e., the heteroclinic tangency occurs.

(c) For $R_1>R_c$, $M(t_0, \theta_0)$ has the simple zeros, i.e., the transversal intersection of W^s and W^u occurs.

Proposition 1 leads the following conjecture: When the angular frequency difference between the signal and the interference is small but finite (i.e., $\nu_1=O(1)$) and the PLL is low damping (i.e., $1/\sqrt{K_0\tau}=O(\varepsilon)$), complex dynamics such as long transients to *lock-in* and the fractal basin boundary might appear, due to the transversal intersection of the system-generated invariant manifolds.

3.3 Numerical Results

To verify the validity of the above conjecture in the read problem, we performed the numerical integration

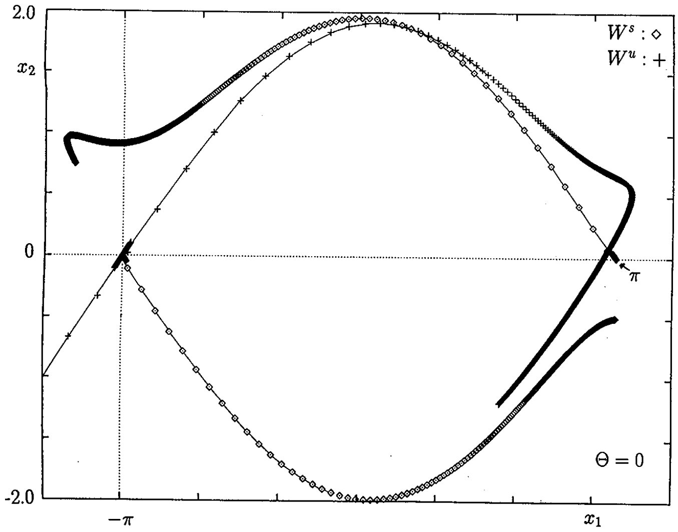

Fig. 4 Numerically obtained invariant manifolds W^s and W^u for a practical set of parameters: $1/\sqrt{K_0\tau}=0.01$, $R_1=0.08$ and $\nu_1=0.8$.

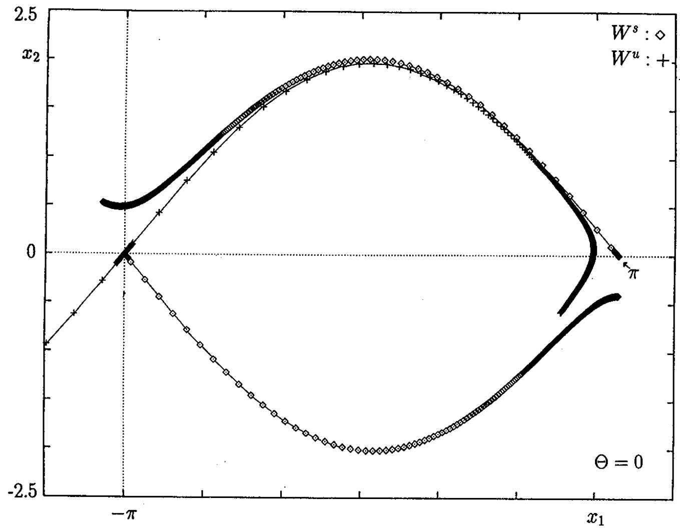

Fig. 5 Numerically obtained W^s and W^u for $1/\sqrt{K_0\tau}=0.01$, $\nu_1=0.8$ and $R_1=0.01$.

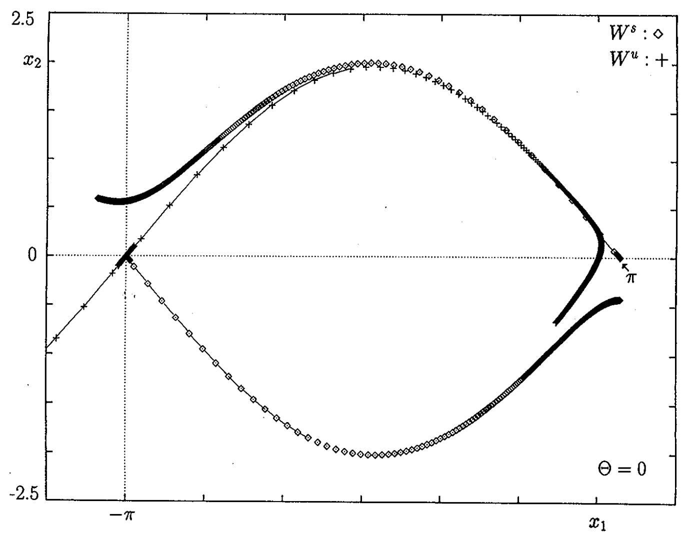

Fig. 6 Numerically obtained W^s and W^u for $1/\sqrt{K_0\tau}=0.01$, $\nu_1=0.8$ and $R_1=0.02$.

of the system-generated invariant manifolds W^s and W^u for a practical set of parameters. We used the 4th order Runge-Kutta integration scheme with integration step at 0.01. 1400 initial points on the Poincaré section $\Theta_1=0$ were set respectively along the eigenvectors associated with the stable and the unstable manifolds from a hyperbolic periodic orbit. The result in Fig. 4 confirms the transversal intersection of W^s and W^u on the section $\Theta_1=0$, for a practical set of parameters $1/\sqrt{K_0\tau}=0.01$, $R_1=0.08$, and $\nu_1=0.8$. The result in Fig. 5 shows that W^s and W^u are close but separate, while in Fig. 6 the intersection of W^s and W^u can be identified. Hence, the heteroclinic tangency is suggested to occur between $R_1=0.01$ and $R_1=0.02$ with $1/\sqrt{K_0\tau}=0.01$ and $\nu_1=0.8$. The theoretical value of the critical parameter R_c is obtained from the equation

$$-\frac{8}{K_0\tau}+\pi R_1\left\{\frac{2(1-\nu_1^2)}{\cosh(\pi\nu_1/2)}-\frac{2\nu_1^2}{\sinh(\pi\nu_1/2)}-2\sinh(\pi\nu_1/2)\right\}=0, \tag{17}$$

with $1/\sqrt{K_0\tau}=0.01$ and $\nu_1=0.8$, as $R_c=0.006990$. The discrepancy between the theoretical and experimental value of R_c is considered to due the finite value of $1/\sqrt{K_0\tau}$.

4. Conclusions

In summary, this letter reports the analytic and numerical study on the geometric structure of the invariant manifolds in PLL in the presence of a weak CW interference. The transversal intersection of W^s and W^u is proven to occur under some coditions on the system parameters via the Melnikov integral, while the existence of the transversal intersection of these invariant manifolds was confirmed numerically for a practical set of parameters. These results are considered to provide an understanding of the complex phenomena onset of PLL in the presence of a weak CW interference. Although the Melnikov homoclinicity condition does not guarantee the existence of a chaotic attractor, complex dynamical properties such as the fractal basin boundary or extremely long transients is expected to occur. These will be discussed in the near future.

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